4.9 - Solving Systems of Linear DEs by Elimination

The DE $y' + 2y + 3x = -1$ can be written in terms of the differential operator D, where D represents a derivative. In this form, the DE becomes $Dy + 2y + 3x = -1$. This can also be written $(D+2)y + 3x = -1$. Our work in this lesson makes use of this notation. For this lesson, x and y are functions of t, so $Dx = \frac{dx}{dt}$, $Dy = \frac{dy}{dt}$, and $D^2x = \frac{d^2x}{dt^2}$, etc.

Solve the given system of differential equations by elimination.

Example:

$$
\begin{aligned}
① \quad (D+1)x &+ (D-1)y = 2 \\
② \quad 3x &+ (D+2)y = -1
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
x' + x + y' - y &= 2 \\
3x + y' + 2y &= -1
\end{aligned}
$$

$3 \cdot ① - (D+1)②$:

$$
\begin{aligned}
\cancel{3(D+1)x} + 3(D-1)y &= 6 \\
\cancel{-3(D+1)x} - (D+1)(D+2)y &= -(D+1)(-1)
\end{aligned}
$$

$-D(-1) - 1(-1)$

$-\frac{d}{dt}(-1)$

$$[(3D-3) - (D+1)(D+2)]y = 6 + 1$$

$$(\cancel{3D} - 3 - D^2 - \cancel{3D} - 2)\, y = 7$$

$$(D^2 + 5)\, y = -7 \quad \Rightarrow \quad y'' + 5y = -7$$

$$m^2 + 5 = 0 \Rightarrow m = \pm\sqrt{5}\, i$$

$$y_c = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t \qquad y_p = A$$

$$y = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t - \frac{7}{5} \qquad A = -\frac{7}{5}$$

②: $x = -\frac{1}{3} - \frac{1}{3}(D+2)y$

$$Dy = -c_1\sqrt{5}\sin\sqrt{5}t + c_2\sqrt{5}\cos\sqrt{5}t$$

$$x = -\frac{1}{3} + \frac{c_1\sqrt{5}}{3}\sin\sqrt{5}t - \frac{c_2\sqrt{5}}{3}\cos\sqrt{5}t$$

$$- \frac{2c_1}{3}\cos\sqrt{5}t - \frac{2c_2}{3}\sin\sqrt{5}t + \frac{14}{15}$$

$$\boxed{\begin{aligned} x(t) &= \left(-\tfrac{2}{3}c_1 - \tfrac{\sqrt{5}}{3}c_2\right)\cos\sqrt{5}t + \left(\tfrac{\sqrt{5}}{3}c_1 - \tfrac{2}{3}c_2\right)\sin\sqrt{5}t + \tfrac{3}{5} \\ y(t) &= c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \tfrac{2}{5} \end{aligned}}$$

Alternative after finding $y(t)$.

Eliminate y:

$$\begin{aligned} ① \quad (D+1)x &+ (D-1)y = 2 \\ ② \quad 3x &+ (D+2)y = -1 \end{aligned}$$

$(D+2)① - (D-1)②$

$$\begin{aligned} &(D+2)(D+1)x - \cancel{(D+2)(D-1)y} = (D+2)2 \\ - \; &3(D-1)x - \cancel{(D+2)(D-1)y} = -(D-1)(-1) \end{aligned}$$

$$(D^2 + \cancel{3D} + 2 - \cancel{3D} + 3)x = 3$$

$$(D^2+5)x = 3 \quad \to m^2 + 5 = 0$$

$$x_c = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t \qquad m = \pm\sqrt{5}i$$

$$x(t) = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t + \tfrac{3}{5}$$

$x_p = A$

$A = \frac{3}{5}$

$$y(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \tfrac{2}{5}$$

We have too many constants. x, y are coupled in the system, so we need to connect c_1, c_2 with c_3, c_4

$$\begin{aligned} ① \quad (D+1)x &+ (D-1)y = 2 \\ ② \quad 3x &+ (D+2)y = -1 \end{aligned}$$

②: $3c_3\cos\sqrt{5}t + 3c_4\sin\sqrt{5}t + \frac{9}{5} - c_1\sqrt{5}\sin\sqrt{5}t$
$+ c_2\sqrt{5}\cos\sqrt{5}t + 2c_1\cos\sqrt{5}t + 2c_2\sin\sqrt{5}t$
$- \frac{14}{5} = -1$

cosine: $3c_3 + c_2\sqrt{5} + 2c_1 = 0$ Const: $\frac{9}{5} - \frac{14}{5} = -1$ ✓

sine: $3c_4 - c_1\sqrt{5} + 2c_2 = 0$

$c_3 = -\frac{2}{3}c_1 - \frac{\sqrt{5}}{3}c_2$, $c_4 = \frac{\sqrt{5}}{3}c_1 - \frac{2}{3}c_2$

Example:
$$\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} &= e^t \\ -\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + y &= 0 \end{aligned}$$

$\rightarrow$ ① $Dx + Dy = e^t$
② $(-D^2 + D + 1)x + y = 0$

Elim y: ① $-$ D② $\quad Dx + Dy = e^t$
$$(D^3 - D^2 - D)x - Dy = 0$$
$$(D^3 - D^2)x = e^t \qquad m^3 - m^2 = 0$$

$x_c = c_1 + c_2 t + c_3 e^t \qquad m^2(m-1) = 0$

$x_p = Ate^t \Rightarrow x_p' = (A + At)e^t$

$x_p'' = (2A + At)e^t$, $x_p''' = (3A + At)e^t$

$3A + \cancel{At} - 2A - \cancel{At} = 1 \Rightarrow A = 1$

$$\boxed{\begin{aligned} x(t) &= c_1 + c_2 t + c_3 e^t + te^t \\ y(t) &= -c_1 - c_2 - c_2 t - c_3 e^t - te^t + e^t \end{aligned}}$$

using ②

$3\times 3 \to 2\times 2 \to 1$

soln #2

Example:

$$\begin{aligned}
&① & Dx & & +z &= e^t\\
&② & (D-1)x &+Dy &+Dz &= 0\\
&③ & x &+2y &+Dz &= e^t
\end{aligned}$$

Elim y: $2②-D③$

$$\begin{aligned}
(2D-2)x + \cancel{2Dy} + 2Dz &= 0\\
-Dx - \cancel{2Dy} - D^2 z &= -e^t
\end{aligned}$$

$$④\ (D-2)x + (-D^2+2D)z = -e^t$$

$$①\ Dx + z = e^t$$

Elim x: $D④ - (D-2)①$

$(D-2)(-e^t) = -e^t + 2e^t$

$$\begin{aligned}
\cancel{D(D-2)x} + (-D^3+2D^2)z &= -e^t\\
-\cancel{D(D-2)x} + (-D+2)z &= e^t
\end{aligned}$$

$$(-D^3+2D^2-D+2)z = 0$$

$$m^3-2m^2+m-2=0 \Rightarrow (m^2+1)(m-2)=0$$

$$z(t) = c_1 e^{2t} + c_2\cos t + c_3 \sin t$$

elim z: ④ + (D^2-2D)①

$$(D-2)x + (\cancel{-D^2+2D})z = -e^t$$

$$(D^3-2D^2)x + (\cancel{D^2-2D})z = -e^t$$

$$(D^3-2D^2+D-2)x = -2e^t \Rightarrow (m^2+1)(m-2) = 0$$

$$x_c = c_4e^{2t} + c_5\cos t + c_6\sin t, \quad x_p = Ae^t$$

$$x_p: A - 2A + A - 2A = -2 \Rightarrow A = 1$$

$$x = c_4e^{2t} + c_5\cos t + c_6\sin t + e^t \qquad ①: Dx + z = e^t$$

$$2c_4e^{2t} - c_5\sin t + c_6\cos t + \cancel{e^t} + c_1e^{2t} + c_2\cos t + c_3\sin t = \cancel{e^t}$$

$$c_4 = -\tfrac{1}{2}c_1, \quad c_5 = c_3, \quad c_6 = -c_2$$

$$x(t) = -\tfrac{1}{2}c_1e^{2t} + c_3\cos t - c_2\sin t + e^t$$

$$y(t) = -\tfrac{3}{4}c_1e^{2t} - c_3\cos t\ c_2\sin t - \tfrac{1}{2}e^t$$

$$z(t) = c_1e^{2t} + c_2\cos t + c_3\sin t$$

Check this later.

$$③: y = -\tfrac{1}{2}x - \tfrac{1}{2}Dz + \tfrac{1}{2}e^t$$

$$\tfrac{1}{4}c_1e^{2t} - \tfrac{1}{2}c_3\cos t + \tfrac{1}{2}c_2\sin t - \tfrac{1}{2}e^t$$

$$-c_1e^{2t} + \tfrac{1}{2}c_2\sin t - \tfrac{1}{2}c_3\cos t$$

Example:

$$\frac{dx}{dt} = -x \quad +z$$
$$\frac{dy}{dt} = -y \quad +z$$
$$\frac{dz}{dt} = -x \quad +y$$